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Analyticity of Low-Temperature Phase Diagrams of Lattice Spin Models

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The analyticity of all strata of the Pirogov–Sinai phase diagram is proved. As a byproduct of the method, a characterization of typical volumes for which the complex partition function vanishes is given, for a Hamiltonian that is a perturbation of the real-valued one, near the point of a phase transition.

KEY WORDS: Pirogov–Sinai theory; contour models; large and small contours; complex contour functionals; analyticity of the phase diagram; localization of zeros of partition functions.

1. INTRODUCTION

The aim of this paper is to present a method enabling one to prove the analyticity of all the strata of the Pirogov–Sinai phase diagram. The models we study are the discrete spin models of the Pirogov–Sinai theory[1] and our approach is based on Ref. 2.

We start with a few comments on the relation between this paper and related work[3,4]: While Ref. 4 uses some ideas not very different from ours, it has a distinguishing feature—the use of the block spin techniques. That work and the present paper were developed independently.

The first proof of analyticity of the diagram is due to Basuev.[3] His method, based on the use of Mayer-type expansions (as developed in his earlier papers), is different from ours and this stimulated us to write down our proof (announced in Ref. 2). Moreover, we expect that our method also will work without any substantial changes in the case of continuum spin models. The supplementary technique needed to control the continuum spin models is developed in detail in Ref. 5. (It also can be used to study interfaces.[6]) We do not formulate it here, nor do we formulate any analyticity properties of the interface diagram in Ref. 7. [Since the ν-dimen-

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sional interface problem is formulated in Ref. 7 in terms of the problem of characterization of the translation-invariant phases of some auxiliary $(\nu-1)$-dimensional Pirogov–Sinai (PS) situation, our results here can be applied to Ref. 7.]

The characteristic feature of Ref. 2 and our approach here is the notion of "small" and "large" contours used to distinguish between the "stable" and "possibly unstable" behaviour of the system. Under "stable" boundary conditions, all contours are "small." "Large" contours may appear under "unstable" boundary conditions (for large volumes only); they are defined as boundaries of the large droplets of stable phase (which "tend to appear inevitably" in those large volumes).

The problems one has to tackle in the PS theory when going to the complex Hamiltonians can be characterized as follows. One has to generalize the following two kinds of arguments:

(i) The technique of contour models (developed primarily for the description of the coexisting "stable" phases; some cluster expansions of the partition functions of the contour models are needed here.

(ii) The inequalities controlling the behavior of partition functions of the "nonstable phases" compared to the stable ones. These inequalities are also based on (i) but require some additional considerations.

It is instructive to notice that the problem is not in the cluster expansions (they work also in the complex case), but in the estimates (ii). Namely, some estimates from below (especially of the partition functions corresponding to "nonstable phases") deteriorate as the terms forming the partition functions are no longer positive! Thus, our strategy is to adapt the method of Ref. 2 avoiding or modifying the estimates from below of the complex partition functions.

This is related to the question of zeros of the partition functions. That they do appear is a trivial consequence of the very existence of phase transitions. Here we obtain, as a byproduct of our method, some bounds on the localization of the zeros "nearest to the real axis." Moreover, we explain the very "mechanism" of the phenomenon, finding (with a good accuracy) for a given complex Hamiltonian a "critical" volume Λ such that $Z(\Lambda)=0$. This is formulated in Theorem 2, for the "diluted" partition functions. Roughly speaking, e.g., for the Ising model with an external field $h>0$, the "diluted" partition function $Z^+(\Lambda)$ corresponding to the $(+)$ boundary condition attains the value 0 for a suitable set (almost a cube) Λ of the size Ch^{-1}, assuming that the external field changes to $h+C'Th^\nu$ (T is the temperature) for a suitable complex C'. This will be shown to be closely related to the phenomenon that Λ is a "critical" set in the following sense: For some other value of the field $h+C''h^\nu$, C'' real, the configurations in Λ containing a large droplet of the $(-)$ phase tend to prevail in their contribution to $Z^{+}(\Lambda)$ compared to the contribution of all "metastable configurations persisting in the $+$ regime."

These results show the special role of the periodic, resp. empty, boundary conditions in the Lee–Yang theorem (where zeros appear only for $\operatorname{Re} h = 0$).

Theorem 2 also complements our main result, Theorem 1, by showing the limits of what can be proved by this method. (Concerning the non-existence of the analytic continuation of the thermodynamic quantities outside the given strata of the phase diagram, one has to refer to the deep and valuable Isakov method.[9])

Theorem 1 of the present paper was announced in Ref. 10. Here, we prove it in Section 2.2. First, in Section 2.1, we briefly recapitulate the setting of the problem and the basic method of Ref. 2. This is done for the convenience of the reader. At the same time, it enables us to introduce one important modification of the strategy of Ref. 2: The notion of a small contour depends on the Hamiltonian in Ref. 2. In order to guarantee the continuity of various quantities such as the free energies of the "metastable" models, the notion of a "truncated" functional $\max(F(\Gamma), \frac{1}{3}\tau\,|\operatorname{supp}\Gamma|)$ was introduced in Ref. 2. It is clear that such an operation is not reasonable when the proof of analyticity is required.

Here, we adopt the following strategy. Starting with some real Hamiltonian $H(\lambda_0)$, we define the notion of a small contour as in Ref. 2, strengthen it further a little bit for the contours with "unstable" exterior (excluding contours with too large length), and then *fix* this notion in some neighborhood of λ_0. Quantities such as s_q in Ref. 2 are now defined as the free energies of the contour models admitting small contours only. With these modifications, Refs. 2 and 7 can be consulted for more details and some related information.

The limitation of our method (as compared to Ref. 4) is that it works only "near the real axis," assuming that the complex Hamiltonian is a small perturbation of a real one. (Only in the case when the phases coexist does our method also work everywhere; it is, in fact, reduced to the method of Ref. 8.)

2. THE PHASE DIAGRAM

2.1. The Background Situation of the Real Hamiltonians

In this section we recapitulate the basic notions of Ref. 2 (see also Refs. 7 and 10). We will explain here our basic technique, starting with the case of a real Hamiltonian.

We consider a configuration space

$$X = S^{\mathbb{Z}^\nu}$$

with S finite, $\nu \geqslant 2$. The Hamiltonian is given as

$$H(x) = \sum \Phi_A(x_A) \tag{1}$$

with translation-invariant interactions Φ_A, diam $A \leqslant r$. The models we consider are typically the low-temperature ones. We will incorporate the inverse temperature into H, i.e., temperature will be one of the parameters in the Hamiltonian.

Given a finite family $\{x^q, q \in Q\}$ of constant configurations ("ground states of the unperturbed Hamiltonian"), we define the notion of a q-contour Γ^q in the usual way.[1,2,7] If all the contours Γ_i of some configuration x satisfy the condition either supp $\Gamma_i \subset \Lambda$ or supp $\Gamma_{i'} \subset \Lambda^c$, then considering the restriction x_Λ (of x to Λ), it is useful to write its Hamiltonian as[2,7]

$$H(x_\Lambda) = \sum_i \Phi(\Gamma_i) + \sum_{q \in Q} e_q \, |\Lambda_q| \tag{2}$$

where $\Phi(\Gamma)$ are suitable translation-invariant "contour Hamiltonians," e_q is the density of energy of x^q, and Λ_q denotes the set of all $t \in \Lambda$ that are either "q-correct" or belong to some supp Γ_i^q.

The convenient property of (2) is its additivity:

$$H(x_\Lambda \cup x_{\Lambda'}) = H(x_\Lambda) + H(x_{\Lambda'}) \qquad \text{if} \quad \Lambda \cap \Lambda' = \varnothing$$

We emphasize that H will be written in the form (2) everywhere. No Hamiltonians defined with respect to general boundary conditions will be used (with one slight exception in Section 3.1).

A configuration x_Λ is called q-diluted if for any contour Γ of x_Λ, dist((supp $\Gamma \cup$ int Γ), $\Lambda^c) \geqslant 2$ and if x_Λ is equal to q on the boundary of Λ. Hence, one has the notion of a diluted partition function

$$Z^q(\Lambda) = \sum_{\text{all } q\text{-diluted } x_\Lambda} \exp[-H(x_\Lambda)]$$

Another "traditional" notion is that of a "crystallic" partition function in the volume $V(\Gamma) = \text{supp}\, \Gamma \cup \text{int}\, \Gamma$:

$$Z(\Gamma) = \sum \exp[-H(x_\Gamma)]$$

where the sum is taken over all configurations x_Γ on $V(\Gamma)$, which can be extended to $\mathbb{Z}^\nu$ such that Γ is a contour of the extended configuration.

We will compare $Z(\Gamma)$ with the "reference" partition function (called "reduced" in Refs. 2 and 7)

$$Z_{\text{ref}}(\Gamma) = \sum \exp[-H(x_\Gamma)]$$

the sum being taken over all configurations x_Γ on $V(\Gamma)$ that are equal to q on supp Γ and are q-diluted on int Γ, assuming that $\Gamma = \Gamma^q$, i.e., that q is the "exterior color" of Γ.

The basic notion of a contour functional $F(\Gamma)$ is defined from the relation

$$Z(\Gamma) = \exp[-F(\Gamma)]\, Z_{\text{ref}}(\Gamma) \tag{3}$$

The *contour model* (more precisely, q-contour model) is then defined as a polymer model with partition functions

$$Z_\Lambda^q = \sum \prod_i k_{\Gamma_i^q}$$

the sum being taken over all families of q-contours such that $\text{dist}(\text{supp}\,\Gamma_i, \Lambda^c) \geqslant 2$ and $\text{dist}(\text{supp}\,\Gamma_i, \text{supp}\,\Gamma_{i'}) \geqslant 2$. The "activity" k_Γ is given as

$$k_\Gamma = \exp[-F(\Gamma)]$$

It can be shown that the definition (3) implies the *equivalence* of both the "physical" and the contour ensembles: namely, for any Λ with simply connected components we have

$$Z^q(\Lambda) = \exp(-e_q\,|\Lambda|)\, Z_\Lambda^q$$

It turns out from this relation that the behavior of external contours of q-diluted configurations in Λ is the same as in the q-contour model.

The contour models can be studied (under conditions specified below) by using the method of cluster expansions (see, e.g., Ref. 7). A reasonable assumption guaranteeing "good behavior" of these cluster expansions is

$$F(\Gamma) \geqslant \tau\, |\text{supp}\,\Gamma| \qquad \text{with } \tau \text{ large} \tag{4}$$

for all contours of the given q-contour model.

In many models of the type (1) possessing only a finite number of ground states we have *a priori* given the analogous condition for $\Phi(\Gamma)$, called the Peierls (or Gertzik–Pirogov–Sinai) condition[1,7]: For all Γ,

$$\Phi(\Gamma) \geqslant \tau\, |\text{supp}\,\Gamma| \qquad \text{with } \tau \text{ large} \tag{5}$$

This condition will be *assumed to hold* in the following. It is instructive to notice that even in this case, the condition (4) can be violated for some Γ^q. This indicates the "nonstability of q," implying in fact that $F(\Gamma)\,|\text{supp}\,\Gamma|^{-1}$ is almost zero for a suitably large Γ^q.

Such contour models cannot be studied by the usual methods of cluster expansions and so the notion of a contour model must be supplemented by further devices. Pirogov and Sinai[1] use the contour models with a parameter. Here we use the alternative approach of Ref. 2:

Definition. A contour Γ^q is called *stable* if

$$F(\Gamma^q) \geqslant \tfrac{1}{3}\tau\,|\text{supp}\,\Gamma| \tag{6}$$

A (stable) contour Γ^q satisfying the condition that even any $\tilde{\Gamma}^q$, supp $\tilde{\Gamma}^q \subset V(\Gamma^q)$ is stable is called a *small* contour. Denote by $-s_q$ the free energy of the "metastable" contour model defined as the polymer model with $k_{\Gamma^q} = \exp[-F(\Gamma^q)]$, Γ^q small. Write

$$h_q = e_q - s_q, \qquad h = \min\{h_q\}, \qquad a_q = h_q - h$$

We say that q is *stable* if $a_q = 0$. (See Note 1 below Theorem 1 for the interpretation of the stability of q).

Note. There is some "freedom" in the definition of a stable (correspondingly, small) contour. In fact, it will be useful to modify this definition in Section 2.2 and again in Section 3.1, adapting it to the particular problem. Still another definition is used in Ref. 7! The notion of a stable q and the value of h is of course unaffected by these modifications. h is the free energy.

Theorem 0.[2]

(i) Nonstable contours satisfy the inequality

$$a_q\,|\text{int}\,\Gamma^q| \geqslant \tfrac{1}{3}\tau\,|\text{supp}\,\Gamma^q| \tag{7}$$

In particular, there are no unstable Γ^q for q stable.

(ii) $$Z^q(\Lambda) \geqslant \exp(-h_q\,|\Lambda| - C\,|\partial\Lambda^c|) \tag{8}$$

(iii) $$Z^q(\Lambda) \leqslant \exp(-h\,|\Lambda| + C\,|\partial\Lambda^c|) \tag{9}$$

where $C = C(\tau)$ is some constant such that $\lim_{\tau\to\infty} C = 0$.

Proof. See Ref. 2. The important preliminary step of the proof is an estimate of the polymer partition function Z^q_Λ:

$$|\log Z^q_\Lambda - s_q\,|\Lambda|| \leqslant C\,|\partial\Lambda| \tag{10}$$

(or $C\,|\partial\Lambda^c|$, which is often a more suitable formulation), where $C = C(\tau) \to 0$ as $\tau \to \infty$. The quantity $-s_q$ denotes the free energy of the contour model (see also Ref. 7).

Such an estimate holds for any polymer model satisfying a condition of the type $|k_\Gamma| \leqslant \varepsilon^{|\operatorname{supp}\Gamma|}$ with small ε. More precisely, the relation (10) typically will be used for the polymer partition function $Z_\Lambda^{q\,\mathrm{small}}$ corresponding to all possible collections of small contours only [and so satisfying the condition (6)].

The proof of (ii) follows easily from (10) if we apply it to the right-hand side of the obvious inequality

$$Z^q(\Lambda) = \exp(-e_q\,|\Lambda|)\,Z_\Lambda^q \geqslant \exp(-e_q\,|\Lambda|)\,Z_\Lambda^{q\,\mathrm{small}}$$

The proof of (i) and (iii) is given by induction over the "level" of Γ, resp. Λ. While the induction step for (i) is a straightforward combination of (ii), (iii), and the definition of nonstable contour, the proof of (iii) uses some considerations to be repeatedly used later and so we present it here: Consider, for brevity, the case $Q = \{+, -\}$, $h_- = h = h_+ - a$ with $a > 0$. Estimate $Z^+(\Lambda)$: Fix a family of all external *large* contours (of some configuration). Given such a family $\{\Gamma_i\}$, the corresponding partition function $Z^+(\Lambda, \{\Gamma_i\})$ is smaller than

$$\exp\left\{-h_+\,|\mathrm{ext}| - h\,|\mathrm{int}| - \sum_i \left[\Phi(\Gamma_i) + e_+\,|\operatorname{supp}\Gamma_i|\right] + C\left(|\partial\Lambda^c| + \sum_i |\operatorname{supp}\Gamma_i|\right)\right\}$$

(see Fig. 1); we sum over all diluted configurations in $\mathrm{ext} = \Lambda \setminus \bigcup_i V(\Gamma_i)$ with small contours only. This yields the term $\exp(-h_+\,|\mathrm{ext}| \pm C\,|\mathrm{ext}|)$ by

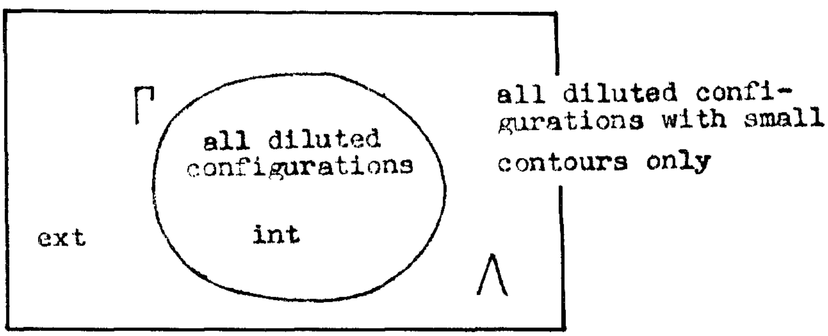

Fig. 1

(10). Similarly, we sum over all diluted configurations in $\mathrm{int} = \bigcup_i \mathrm{int}\, \Gamma_i$ and use the inductive assumption (iii) for this sum.

Summing over all possible $\{\Gamma_i\}$, changing slightly Φ,

$$\Phi'(\Gamma) = \Phi(\Gamma) - C' \, |\mathrm{supp}\, \Gamma|$$

we have

$$Z^+(\Lambda) \leqslant \exp(-h\, |\Lambda| + C\, |\partial \Lambda^c|) \sum_{\{\Gamma_i\}} \exp\left[-a\, |\mathrm{ext}| - \sum_i \Phi'(\Gamma_i) \right] \tag{11}$$

We have rewritten the problem such that it fits the following scheme:

Lemma 1 ("Model of unstable behavior"; main lemma of Ref. 2; we formulate its complex variant, since this also will be used later). Consider a model where "configuration" means a collection of mutually external contours Γ_i such that $\mathrm{dist}(V(\Gamma_i), \Lambda^c) \geqslant 2$ and $\mathrm{dist}(\mathrm{supp}\, \Gamma_i, \mathrm{supp}\, \Gamma_{i'}) \geqslant 2$, $i \neq i'$. The Hamiltonian is given as

$$H\{\Gamma_i\} = a\, |\mathrm{ext}| + \sum_i \Phi(\Gamma_i) \qquad \text{where} \quad \mathrm{ext} = \Lambda \Big\backslash \bigcup_i V(\Gamma_i)$$

with some complex a, $\Phi(\Gamma)$. Denote by

$$Z(\Lambda) = \sum \exp(-H\{\Gamma_i\})$$

where the sum is overall possible choices of $\{\Gamma_i\}$. Let $\mathrm{Re}\, \Phi(\Gamma) \geqslant \tau\, |\mathrm{supp}\, \Gamma|$ for all Γ. Then for some $C = C(\tau)$ such that $\lim_{\tau \to \infty} C = 0$ the following holds: Consider the polymer model with activities

$$k_\Gamma = \exp[-\mathrm{Re}\, \Phi(\Gamma) + 2C\, |\mathrm{supp}\, \Gamma|]$$

Denote by $-s$ its free energy and assume that

$$s < \mathrm{Re}\, a \tag{12}$$

Then, for τ large enough,

$$|Z(\Lambda)| \leqslant \exp(C\, |\partial \Lambda^c|) \tag{13}$$

Notes. 1. By substituting (13) into (11), the proof of (iii) can be concluded. To this end, it suffices to check the condition (12) for the polymer model with $k_\Gamma = \exp[-\Phi'(\Gamma) + 2C\, |\mathrm{supp}\, \Gamma|]$, Γ large. Actually, it can be shown[2] that s behaves like $\exp[-C''(\tau a^{-1})^{\nu - 1}]$, which is surely $\ll a$ for large τ.

2. In the general case of $Q \neq \{+, -\}$, one replaces Lemma 1 by its generalization from Section 2 of Ref. 2.

Proof of Lemma 1. Denote by Z_Λ the partition function of the polymer model with the activity k_Γ. Given any collection of external contours $\{\Gamma_i\}$ contributing to Z_Λ, "fill in" $M = \bigcup_i \operatorname{int} \Gamma_i$ by another contour, "inserting the term" $Z_M \exp(-s\,|M|)$, which is approximately equal to 1. Writing $\bar{a} = \operatorname{Re} a$ and $\bar{\Phi} = \operatorname{Re} \Phi$, we have the estimate

$$\begin{aligned}
|Z(\Lambda)| &\leqslant \sum_{\{\Gamma_i\}} \exp(-\bar{a}\,|\text{ext}|) \prod_i \exp[-\bar{\Phi}(\Gamma_i)] \\
&\leqslant \sum_{\{\Gamma_i\}} \exp(-\bar{a}\,|\text{ext}|) \\
&\quad \times \prod_i \exp[-\bar{\Phi}(\Gamma_i)]\, Z_M \exp(-s\,|M| + C\,|\partial M^c|) \\
&\leqslant \exp(-s\,|\Lambda|)\, Z_\Lambda
\end{aligned}$$

(notice that $s < \bar{a}$ and $C \sum_i |\operatorname{supp} \Gamma_i| > C\,|\partial M^c|$; we can also assume that $C > s$)

$$\leqslant \exp(C\,|\partial \Lambda^c|)$$

2.2. Analyticity of the Phase Diagram

The existence of the phase diagram in the sense of its continuity (more precisely, continuity of all strata with respect to the parameters on which the Hamiltonian continuously depends) is proved in Refs. 1 and 2 and most generally in Ref. 7. It is shown there that the low-temperature phase diagram "mimicks" the zero-temperature one, under some general conditions. Here we are concerned with the proof of the analyticity properties of all strata of this phase diagram.

Let $H = H(\boldsymbol{\lambda})$ [given by (2)] depend on some vector real parameter $\boldsymbol{\lambda}$. Denoting by

$$Q(\boldsymbol{\lambda}) = \{q \in Q\colon q \text{ is stable for } H(\boldsymbol{\lambda})\}$$

we will be interested, for any fixed subset $\bar{Q} \subset Q$, in the behavior of the stratum $\{\boldsymbol{\lambda}\colon Q(\boldsymbol{\lambda}) = \bar{Q}\}$. Writing $\bar{Q} = \{q_1, ..., q_n\}$, we will assume that $\boldsymbol{\lambda}$ can be written in the form

$$\boldsymbol{\lambda} = (\lambda^1, ..., \lambda^{n-1}, \mu^1, ..., \mu^m)$$

where $n \geqslant 2$ and $m \geqslant 1$.

Note. The case $n = 1$ (analyticity of the thermodynamic functions in the uniqueness region) is not studied here. (It requires, in fact, only a part of the technique developed below. The other part is represented by another extremal case, $n = |Q|$.)

Theorem 1. Let λ_0 be such that either (i) $\bar{q} \in \bar{Q} \Leftrightarrow e_{\bar{q}}(\lambda_0) = \min_{q \in Q}\{e_q(\lambda_0)\}$ or (ii) $\bar{Q} = Q(\lambda_0)$.

Assume the invertibility of the matrix

$$\left(\frac{\partial(e_{q_i} - e_{q_1})}{\partial \lambda^j}\right) \quad \begin{matrix} i = 2,..., n \\ j = 1,..., n-1 \end{matrix} \tag{14}$$

(a variant of the degeneracy-removing condition). Assume the real analyticity (and translation invariance) of all the mappings $\{\lambda \rightsquigarrow \Phi_A^\lambda(x_A)\}$ in some neighborhood ϑ of λ_0. Assume that for the Hamiltonian $H(\lambda_0)$ the Peierls condition (5) holds, with a suitably large τ. Then there is some neighborhood $\tilde{\vartheta}$ of λ_0 and a real analytic function

$$\{\boldsymbol{\mu} = (\mu^1,..., \mu^m) \rightsquigarrow (\lambda^1,..., \lambda^{n-1})\}$$

such that its graph intersects $\tilde{\vartheta}$ [case (i)] or even contains λ_0 [case (ii)] and such that

$$\{(\lambda^1(\boldsymbol{\mu}),..., \lambda^{n-1}(\boldsymbol{\mu}), \mu^1,..., \mu^m)\} \cap \tilde{\vartheta} = \{\lambda \in \tilde{\vartheta} : \bar{Q} = Q(\lambda)\}$$

Notes. 1. In Ref. 7 a more detailed interpretation of the notion of a stable $q \in Q$ is given in terms of the existence of a Gibbs state that is a small perturbation of the "qth ground state" x^q. No other translation-invariant Gibbs states exist.[2]

2. An analogous result can be formulated even when starting from the formulation (2). In that case, some additional bounds on $(d/d\lambda)\, \Phi^\lambda(\Gamma)$ are needed.

Proof of Theorem 1. We consider only case (ii). [In Refs. 1 and 7 it is explained how to find in case (i) some λ in the vicinity of λ_0 such that $Q(\lambda) = \mathbf{Q}$. The condition (14) is used there.] We restrict ourselves to the special (but characteristic) case when $|Q| = 3$ and $|\bar{Q}| = 2$. Write $Q = \{+, -, 0\}$ and $\bar{Q} = \{+, -\}$ with 0 unstable. We will take $\lambda_0 = \mathbf{0}$. The proof in the general case is quite analogous, using instead of Lemma 1 its generalized version from Ref. 2.

The idea of the proof is the following. Expecting that 0 will remain unstable in some neighborhood of $\mathbf{0}$, we have to control the "balance" between $(+)$ and $(-)$. Consider a contour $\Gamma = \Gamma^+$. Writing $\operatorname{int} \Gamma = M^+ \cup M^- \cup M^0$, where $M^+ = \operatorname{int}_+ \Gamma$ [union of the $(+)$ components of $\operatorname{int} \Gamma$], etc., we have the expression

$$Z(\Gamma) = \exp[-\Phi_e(\Gamma)]\, Z^+(M^+)\, Z^-(M^-)\, Z(M^0) \tag{15}$$

where $\Phi_e(\Gamma) = \Phi(\Gamma) + e_q\, |\operatorname{supp} \Gamma|$. Hence

$$F(\Gamma) = \Phi_e(\Gamma) + \log Z^+(M^- \cup M^0) - \log Z^-(M^-) - \log Z^0(M^0) \tag{15'}$$

For $H = H(\mathbf{0})$ this can be written as follows [using (ii) and (iii) of Theorem 0]:

$$F(\Gamma) = \Phi(\Gamma) + \tilde{\Delta} \tag{16}$$

with

$$\tilde{\Delta} \geqslant -\tilde{C}\, |\operatorname{supp} \Gamma|, \qquad \lim_{\tau \to \infty} \tilde{C} = 0 \tag{17}$$

One can express $\tilde{\Delta}$ in terms of the contour functionals $F(\Gamma)$ and look on (16) as an integral equation for F [ignoring the definition (3)]. This is the original PS strategy[1] of finding the contour functionals F and we will use it with one substantial modification concerning the description of the "presumably nonstable" 0 phase. Namely, we will solve (16) as an equation for $F(\Gamma^+)$, $F(\Gamma^-)$ only with $F(\Gamma^0)$ retaining its "physical value" (3). [More specifically, the quantities $F(\Gamma^0)$, Γ^0 small, will be used when expressing the partition functions of the "metastable model." The large contours Γ^0 will be handled by the method of Theorem 0, thus avoiding thus any use of $F(\Gamma^0)$ for them.] We will seek for an analytic solution of (16), with "properly" defined $\tilde{\Delta}$, for *any* $\boldsymbol{\lambda}$. Of course, it will be impossible to retain the full equivalence of (15) and (16) if we want (17) to hold at the same time, which is needed if we want to have contour models satisfying (6).

What we will do is to require this equivalence in the particular case when the free energies $-s_\pm$ of the $(\pm)$ contour model given by $F(\Gamma^\pm)$ satisfy the relation $e_+ - s_+ = e_- - s_-$. This will be the "physical" part of the "formal" solution (16).

Let us now elaborate the details of this strategy. Our first task is to have a reasonable definition of $\tilde{\Delta}$ for all $\boldsymbol{\lambda} \in \vartheta$, where ϑ is some complex neighborhood of $\mathbf{0}$. Put

$$\tilde{\Delta} = \Delta^+_{\text{ref}} - \Delta^-_{M^-} - \Delta^0_{M^0} \tag{18}$$

where

$$\begin{aligned} \Delta^+_{\text{ref}} &= \log Z^+(M^0 \cup M^-) + h_+\, |M^0 \cup M^-| \\ \Delta^-_{M^-} &= \log Z^-(M^-) + h_-\, |M^-| \end{aligned} \tag{19}$$

and analogously for Γ^-. The quantity h_+ (analogously, h_-) is defined as $h_+ = e_+ - s_+$, where $s_+ = \lim |\Lambda|^{-1} \log Z^+_\Lambda$. [Notice that h_+, Δ^+ are expressed as functions of $F(\Gamma^+)$.] The definition of the quantity $\Delta^0_{M^0}$ is slightly more complicated and will be given below.

By (10), the quantities Δ^+_{ref} (analogously $\Delta^-_{M^-}$) satisfy the bounds, for each $\boldsymbol{\lambda}$,

$$|\Delta^+_{\text{ref}}| \leqslant C_+\, |\operatorname{supp} \Gamma| \tag{20}$$

where $C_+ = C_+(\tau) \to 0$ as $\tau \to \infty$, assuming that $F(\Gamma^+)$ [analogously, $F(\Gamma^-)$] satisfies the condition (e.g.)

$$\operatorname{Re} F(\Gamma) \geqslant \tfrac{1}{3}\tau \, |\operatorname{supp} \Gamma| \tag{21}$$

So our task is to define $\Delta^0_{M^0}$ such that $-\Delta^0_{M^0}$ would satisfy a bound of the type (17). Before doing so, notice that if $h_+ = h_-$, $\Delta^0_{M^0}$ should be equal to $\log Z^0(M^0) + h_+ \, |M^0|$ [if (15) and (16) have to be equivalent] and it should satisfy the bound

$$\operatorname{Re} \Delta^0_{M^0} \leqslant C_0 \, |\operatorname{supp} \Gamma| \tag{22}$$

(we cannot expect the two-sided inequality here). Finally, we want an analytical dependence of $\Delta^0_{M^0}$ on $F(\Gamma^\pm)$.

To find a proper definition of $\Delta^0_{M^0}$, write the term $Z^0(M^0)$ in (15) as follows [we expressed $Z^+(\Lambda)$ similarly in the proof of Theorem 0]

$$Z^0(M^0) = \sum_{\{\Gamma_i\}} Z^{\text{small}}(\text{ext}) \exp\left[-\sum_i \Phi_e(\Gamma_i) \right] Z^+(N^+) \, Z^-(N^-) \, Z^0(N^0) \tag{23}$$

where the summation is over all possible collections of mutually external large 0-contours $\{\Gamma_i\}$, Z^{small} means summation over all diluted configurations on $\text{ext} = M^0 \setminus \bigcup_i V(\Gamma_i)$ with small external contours only, and $N^+ = \bigcup_i \operatorname{int}_+ \Gamma_i$ etc. Extracting the "bulk" terms from (23), we arrive at the following result.

Definition. Put $a = h_0 - h_+$, $\Phi_e^+(\Gamma) = \Phi_e(\Gamma) - h_+ \, |\operatorname{supp} \Gamma|$. Define (this is an inductive definition for $N^0 \neq \varnothing$)

$$\begin{aligned} \exp(\Delta^0_{M^0}) = \sum_{\{\Gamma_i\}} Z^{\text{small}}(\text{ext}) \exp\left[h_+ \, |\text{ext}| - \sum_i \Phi_e^+(\Gamma_i) \right] \\ \times \exp(\Delta^+_{N^+} + \Delta^-_{N^-} + \Delta^0_{N^0}) \end{aligned} \tag{24}$$

Note. 1. Starting from (23), all the quantities depend on λ. We will not write this dependence, if there is no ambiguity.

2. $\Delta^0_{M^0}$ is really an analytical function of $F(\Gamma^+)$, $F(\Gamma^-)$. We will see later that there is some arbitrariness in this definition (h_+ could be replaced by h_-, for example).

We will estimate (24) using the method of proof (iii), Theorem 0. This requires the applicability of (10) to Z^{small} for all λ. So we have to modify the notion of a small Γ^0 such that the contour Γ^0 would "remain small" in the sense that, say

$$\operatorname{Re} F(\Gamma, \lambda) \geqslant \tfrac{1}{4}\tau \, |\operatorname{supp} \Gamma| \tag{25}$$

holds for each small Γ and each $\lambda \in \vartheta$, where ϑ is some neighborhood of $\mathbf{0}$.

Definition. Say that a small contour Γ^0 is n-small if diam $\Gamma^0 \leqslant n$ (this is a condition for 0-contours only!).

The integer n used in this definition will be chosen sufficiently large such that the proof of Theorem 0 retains (for $\boldsymbol{\lambda} = \mathbf{0}$) its validity if the notion of n-small contour is imposed instead of the notion of small 0-contour. This means that in Lemma 1 we require an inequality $s_n < a_n$ for the corresponding quantities s_n, a_n. It is, however, obvious that $s_n - s \to 0$ as $n \to \infty$ and $a_n > a$. So $s < a$ implies also $s_n < a_n$ for a sufficiently large n, and this inequality can be extended to some complex $\vartheta \ni \mathbf{0}$. So, we can assume that for each $\boldsymbol{\lambda} \in \vartheta$,

$$|s_n(\boldsymbol{\lambda})| < \operatorname{Re} a_n(\boldsymbol{\lambda}) \tag{26}$$

Returning to (25), we can also guarantee it in some ϑ as

$$|F(\Gamma, \mathbf{0}) - F(\Gamma, \boldsymbol{\lambda})| \leqslant C_n \tag{27}$$

holds for some C_n such that $\lim_{N \to \infty} C_n = 0$ in a suitable $\vartheta \ni \mathbf{0}$.

Note. (27) is a very imprecise bound, but we do not make any attempt to "optimalize the possible radius of ϑ" in our Theorem 1. Some related, more precise bounds will be needed in Section 3.2 (namely Lemma 7).

We emphasize that the notion of an n-small contour [with a suitable n guaranteeing (25), (26)] will be used instead of the notion of small contour everywhere in the following.

Proposition 1. With this modified notion of a small contour, (22) holds for all $\boldsymbol{\lambda} \in \vartheta$ if $\boldsymbol{\Delta}$ is defined by (24).

Proof. Expand the term Z^{small} $(= Z^{n\text{-small}})$ into the volume and boundary term as in (10). Then we obtain from (24) the relation

$$\begin{aligned} \exp(\Delta^0_{M^0}) = \sum_{\{\Gamma_i\}} & \exp[-a\,|\text{ext}| + \tilde{\Delta}_0(\text{ext})] \\ & \times \exp\left[-\sum_i \Phi_e^+(\Gamma_i)\right] \exp(\Delta^+_{N^+} + \Delta^-_{N^-} + \Delta^0_{N^0}) \end{aligned} \tag{28}$$

where $a = a_n(\boldsymbol{\lambda})$, etc., and the summation is over all families of mutually external n-large 0-contours Γ_i in M^0. The term $\tilde{\Delta}_0$ is given as

$$\tilde{\Delta}_0(\text{ext}) = \log Z^{\text{small}}(\text{ext}) + h_0\,|\text{ext}|$$

(the "boundary term" of the n-small 0-contour model).

Investigate first the case $N^0 = \varnothing$ in (24): Write

$$\Phi^*(\Gamma) = \operatorname{Re} \Phi_e^+(\Gamma) - (C' + C'')\,|\text{supp}\,\Gamma|$$

where C' is chosen such that $C' = C'(\tau) \to 0$ if $\tau \to \infty$ and

$$C'\left(\sum_i |\operatorname{supp} \Gamma_i|\right) > |\Delta^+_{N^+}| + |\Delta^-_{N^-}| \tag{29}$$

[recall that there is no problem with establishing the smallness of $\Delta^\pm_{N^\pm} |\partial N^{\pm c}|^{-1}$, since all $F(\Gamma^+)$ and $F(\Gamma^-)$ are assumed to satisfy a condition of the type (21)] and where C'' is chosen analogously, such that

$$C''\left(|\operatorname{supp} \Gamma| + \sum_i |\operatorname{supp} \Gamma_i|\right) \geqslant \tilde{\Delta}_0 \tag{30}$$

[Notice that $(\operatorname{supp} \Gamma + \sum_i |\operatorname{supp} \Gamma_i|) > |\partial \operatorname{ext}^c|$; the smallness of C'' follows now from (25)!] So we can estimate the right-hand side of (28) as

$$\begin{aligned} &\leqslant \exp(C'' |\operatorname{supp} \Gamma|) \sum_{\{\Gamma_i\}} \exp(-a |\operatorname{ext}|) \exp\left[-\sum_i \Phi^*(\Gamma_i)\right] \\ &\leqslant \exp[(C'' + C) |\operatorname{supp} \Gamma|] \end{aligned} \tag{31}$$

by Lemma 1. This is, however, the estimate (22).

Now we are prepared to generalize these estimates to the case when $N^0 \neq \varnothing$ in (24). The only difference, however, is that on the right-hand side of (28) we have a new term $\exp \Delta^0_{M^0}$ and so instead of (29) the inequality

$$C'\left(\sum_i |\operatorname{supp} \Gamma_i|\right) > \Delta^+_{N^+} + \Delta^-_{N^-} + \operatorname{Re} \Delta_{N^0} \tag{32}$$

is needed. For C' suitably larger than $C'' + C$, this surely can be established! So we have the proof of the inductive step in (22) (from N^0 to M^0) with $C_0 = C'' + C$.

We summarize what we have shown:

Corollary. For any $\lambda \in \vartheta$, where ϑ is some complex neighborhood of $\mathbf{0}$, and for any complex contour functionals $F(\Gamma^\pm)$ defined for all contours $\Gamma^\pm$ and satisfying (21), the quantity $\operatorname{Re} \tilde{\Delta}$ defined by (19) satisfies the estimate (17).

Proof. Proposition 1, relation (20), and its analogs for $\Delta^\pm_{M^\pm}$.

So we can formulate the question of solving Eq. (16) for any $\lambda \in \vartheta$, where F is a complex contour functional satisfying (21) for all Γ^+ and Γ^- [and, in general, not connected with the value (3) in any way]. Notice that $\tilde{\Delta}$ was defined as a function of $F(\Gamma^+)$ and $F(\Gamma^-)$ and it also depends on the "physical Hamiltonian" through the values e_q and $F(\Gamma^0)$, Γ^0 n-small.

Denote by $\mathscr{C}$ the set of all functionals $F(\Gamma^{\pm})$ satisfying (21). Define a metric

$$\rho(F, G) = \sup\{|F(\Gamma) - G(\Gamma)| \, |V(\Gamma)|^{-1}\}$$

It is possible to deduce from the expressions (24) and (19) the following estimate:

Proposition. For any real $F, G \in \mathscr{C}$ and any $\lambda \in \vartheta$, λ real,

$$\rho(\tilde{\Delta}(F), \tilde{\Delta}(G)) \leqslant \tilde{\varepsilon}\rho(F, G) \tag{33}$$

with a small $\tilde{\varepsilon}$, $\lim_{\tau \to \infty} \tilde{\varepsilon} = 0$.

The proof of this statement is based on the following general fact from the theory of polymer models.

Lemma 2. Let $k_\Gamma(\lambda)$ be translation-invariant activities depending on some complex parameter λ such that for all Γ,

$$|k_\Gamma(\lambda)| \leqslant \varepsilon^{|\text{supp}\,\Gamma|} \tag{34}$$

$$\left|\frac{d}{d\lambda} k_\Gamma(\lambda)\right| \leqslant \varepsilon^{|\text{supp}\,\Gamma|} \tag{35}$$

with small ε. Then for each Λ,

$$\left|\frac{d}{d\lambda} \log Z_\Lambda\right| \leqslant \tilde{\varepsilon}\,|\Lambda| \tag{36}$$

with small $\tilde{\varepsilon}$. If we denote by $-s$ the free energy of the polymer model, then also

$$\left|\frac{d}{d\lambda} s\right| \leqslant \tilde{\varepsilon} \tag{37}$$

Moreover, denoting $\Delta = \log Z_\Lambda + s\,|\Lambda|$, we also have

$$\left|\frac{d}{d\lambda} \Delta\right| \leqslant \tilde{\varepsilon}\,|\partial\Lambda| \tag{38}$$

Proof of Lemma 2. Condition (36) easily follows from the expression

$$\frac{d}{d\lambda} \log Z_\Lambda = \sum p(\Gamma) \frac{d}{d\lambda} k_\Gamma(\lambda)$$

where $p(\Gamma)=(Z_\Lambda)^{-1} Z_{\Lambda\setminus\operatorname{supp}\Gamma}$, which can be estimated as $|p(\Gamma)|\leqslant (1+\varepsilon)^{|\operatorname{supp}\Gamma|}$. It also immediately implies (37). The relation (38) can be proved from the expression of the type

$$\Delta=\sum_{T\cap\Lambda\neq\varnothing, T\cap\Lambda^c\neq\varnothing} \tilde{k}_T$$

where $\tilde{k}_T$ are given as (see, e.g., Ref. 7 for more details)

$$\tilde{k}_T=\sum\pm\prod_j k_{\Gamma_j}$$

the summation being taken over some distinguished families of $\{\Gamma_j\}$ such that $\bigcup_j \operatorname{supp}\Gamma_j=T$. It is possible to deduce for $\tilde{k}_T$ an inequality of the type $|(d/d\lambda)\,\tilde{k}_T|\leqslant\tilde{\varepsilon}^{|T|}$ with small $\tilde{\varepsilon}$. Then (38) easily follows.

Returning to the proof of the proposition, write $F_\kappa=F+\kappa(G-F)$ and estimate $(d/d\kappa)\tilde{\Delta}$. The estimates of $(d/d\kappa)\,\Delta^{\pm}_{M^\pm}(d/d\kappa)\,\Delta^{\pm}_{\text{ref}}$ now immediately follow from Lemma 2 [in a more precise form (38) than actually needed, even for complex κ]. Concerning $\Delta^0_{M^0}$, the estimate $|(d/d\kappa)\,\Delta^0_{M^0}|\leqslant\tilde{\varepsilon}\,|M^0|$ (with small $\tilde{\varepsilon}$) can be proven by induction, using (24). Namely, for real κ,

$$\frac{d}{d\kappa}\,\Delta^0_{M^0}=\sum_{\mathscr{D}} P(\mathscr{D})\left[\frac{d}{d\kappa}\,(h_+\,|\widetilde{\text{ext}}|+\Delta^+_{N^+}+\Delta^-_{N^-})+\frac{d}{d\kappa}\,\Delta^0_{N^0}\right]$$

where $\widetilde{\text{ext}}=M^0\setminus\bigcup_i \operatorname{int}\Gamma_i$ and where $P(\mathscr{D})$ can be interpreted as the probability that $\mathscr{D}=\{\Gamma_i\}$ occurs [in the ensemble with the partition function (28)]. Applying Lemma 2 to the estimates of the first three terms of this expression, we have

$$\begin{aligned}\left|\frac{d}{d\kappa}\,\Delta^0_{M^0}\right| &\leqslant\sum_{\mathscr{D}} P(\mathscr{D})\,\tilde{\varepsilon}\left(|\text{ext}|+|\partial N^+|+|\partial N^-|+\left|\frac{d}{d\kappa}\,\Delta^0_{N^0}\right|\right)\\ &\leqslant\tilde{\varepsilon}\,|M^0|\end{aligned}$$

if

$$\left|\frac{d}{d\kappa}\,\Delta^0_{N^0}\right|\leqslant\tilde{\varepsilon}\,|N^0|$$

QED. So we have also (33) (with another $\tilde{\varepsilon}$).

Now we can finish the proof of Theorem 1. By the fixed-point theorem, we obtain the limit

$$F(\Gamma,\lambda)=\lim_{n\to\infty} F_n(\Gamma,\lambda) \tag{39}$$

where

$$F_n=F_{n-1}+\tilde{\Delta}(F_{n-1}) \qquad (F_1=\Phi)$$

for any real $\lambda \in \vartheta$. Since we also have the uniform upper bound of the type

$$|\exp[-F_n(\Gamma)]| \leqslant \exp(-\tfrac{1}{3}\tau\,|\text{supp}\,\Gamma|)$$

and since all the functions F_n are obviously holomorphic in λ, we conclude that the convergence in (39) takes place even for complex $\lambda \in \vartheta$. Moreover, F is an analytic function of λ. So $h_\pm(\lambda) = e_\pm(\lambda) - s_\pm(\lambda)$ is an analytic function of λ and we can take the analytic manifold

$$\{\lambda \in \vartheta: h_+(\lambda) = h_-(\lambda)\} \tag{40}$$

Now one has to apply some standard implicit function theorem to deduce the statement (ii) of Theorem 1. We omit the details of this final part of the proof (they can be found in the usual "nonanalytic" formulations of the existence of a phase diagram, e.g., in Refs. 1 and 7) and notice only that the important fact that $(d/d\lambda_i)(h_q - e_q) \leqslant \varepsilon$ with $\lim_{\tau \to \infty} \varepsilon = 0$ is used here, together with the degeneracy-removing condition (14).

3. ZEROS OF DILUTED PARTITION FUNCTIONS

3.1. Critical Volumes. The Case of Real Hamiltonians

For brevity of the proofs we will impose some simplifying assumptions here. Notably, we will consider a special (but characteristic) case $Q = \{+, -\}$ with $+$ nonstable. (The general case can be handled by an analogous method based on the generalized version—Lemma 2.3 of Ref. 2—of our Lemma 1).

Though we will continue to use the convenient formulation (2) of the problem, in one place (Lemma 6) some additional information about the structure of $\Phi(\Gamma)$ will be needed. This will be guaranteed for the models (1) and so we will restrict ourselves to models having a formulation (1), for simplicity. Thus, a typical model we study is the Ising model with a nonzero external field at small temperatures.

For technical reasons we will have to impose the following modification of the notion of a small contour, to be used everywhere in Sections 3.1 and 3.2 (N is again a suitably large integer).

Definition. By a stable contour Γ^+ we mean a contour Γ^+ satisfying the relation diam $\Gamma^+ \leqslant N$ and

$$F(\Gamma^+) \geqslant (\tau/4\nu)\,|\text{supp}\,\Gamma^+| \tag{41}$$

Note. The constant τ is from (5) (we assume that our Hamiltonian satisfies it). The choice of $1/4\nu$ is of course arbitrary, but is related to the choice of $\frac{3}{4}$ in (48). This will impose, requiring the applicability of cluster expansion techniques to the "metastable model," a stronger assumption of largeness of τ than before.

A contour Γ^q is again called small if supp $\tilde{\Gamma}^q \subset V(\Gamma^q)$ for no unstable $\tilde{\Gamma}^q$. This notion will again be *fixed* when working in a neighborhood of a given Hamiltonian.

With this modified notion of a small contour, we can rewrite all the contents of Section 2.1 with changed numerical constants but with the notion of a stable q of course untouched. Write the (new) value of a_+ as $a_+ = a$.

The strategy of Sections 3.1 and 3.2 is the following.

Our main result—Theorem 2—says, roughly speaking, that for a suitable "critical" volume Λ assigned to any given Hamiltonian (near the point of phase transition), a *continuation of the Hamiltonian in a suitable complex direction results in vanishing of the partition function somewhere in the vicinity.*

Section 3.1 is devoted to finding such a critical volume, which is characterized by the following properties:

1. Both the "metastable" configurations having small external contours only and the configurations going to the stable $(-)$ state through some large $(+)$ contour have approximately the same contribution to the partition function $Z^+(\Lambda)$:

$$Z^{+\text{small}}(\Lambda) = Z^{+\text{large}}(\Lambda)$$

2. The large contours appearing in the configurations contributing to Z^{large} are "really large" in the sense that they encircle more than $\frac{3}{4}$ of the volume of Λ.

The latter condition will enable us to distinguish clearly between the behavior of configurations contributing to Z^{small} and Z^{large}: in the former case, a typical value of a typical configuration is $+$ (everywhere in Λ), while in the latter case, more than $\frac{3}{4}$ of all sites of any configuration contributing to Z^{large} are in the $-$ regime.

In Section 3.2 we will show the following: if we change the specific energy (relative to $-$) of $+$ by the value $\pi i\,|\Lambda|^{-1}$, then the quantity $Z_+^{\text{small}}(\Lambda)\,Z_-(\Lambda)^{-1}$ roughly speaking multiplies itself by $\exp(-\pi i) = -1$, while $Z_+^{\text{large}}(\Lambda)\,Z_-(\Lambda)^{-1}$ changes much less significantly. Elaborating this mechanism slightly further, we obtain that $Z_+^{\text{small}} = -Z_+^{\text{large}}$ for a suitable complex field of the approximate intensity $\frac{1}{2}\pi i\,|\Lambda|^{-1}$. Therefore, $Z_+(\Lambda) = 0$.

Let us go back to the contents of the present subsection. Our construction of the set Λ will be based on the idea of a "critical size" of a $+$ contour: This notion will be defined for small $a > 0$ (which is the restriction used everywhere in the following; "almost stability" of $+$). Below the critical size, contours are "undesirable" and one has to pay a great amount of work to install them. [This amount is measured by $F(\Gamma^+)$.] Above the critical size, the situation is just the opposite, for suitably shaped contours [the fact that $F(\Gamma^+)$ drops almost to zero indicates it].

Closely related to the notion of a critical size are the following notions of "the best possible shape of a contour" and the "minimal possible energy" of a contour (given the cardinality of the volume it encircles).

Notation. Denote by $V^*(\Gamma) = \operatorname{supp} \Gamma \cup \operatorname{int}_- \Gamma$ (for $\Gamma = \Gamma^+$). [In fact, this notion is much more important and natural than the conventional $V(\Gamma)$. This is true for all considerations of the PS theory!]

Definition. Denote by $\tilde{\tau}(\Gamma)$ the quantity

$$\tilde{\tau}(\Gamma) = \Phi(\Gamma)[V^*(\Gamma)]^{(1-\nu)/\nu}$$

Denote by

$$\tilde{\tau} = \lim_{n \to \infty} \inf_{\Gamma : |V^*(\Gamma)| = n} \tilde{\tau}(\Gamma)$$

{Typically, $|[V^*(\Gamma)]^{(1-\nu)/\nu}|$ behaves like $|\operatorname{supp} \Gamma|$}.

Definition. Say that a convex subset M of the Euclidean space $\mathbb{R}^\nu$ is an optimal shape of the given Hamiltonian if the following holds: There is some $C > 0$ and some $\eta_\kappa > 0$ such that $\lim_{\kappa \to \infty} \eta_\kappa = 0$ and such that for each $\kappa > 0$ there is some contour $\tilde{\Gamma}_\kappa$ satisfying the following assumptions:

(i) $V^*(\tilde{\Gamma}_\kappa) \subset \kappa M$.

(ii) $\operatorname{dist}(t, \kappa M^c) \leqslant C$ for each $t \in \kappa M \setminus V(\tilde{\Gamma}_\kappa)$.

(iii) $\tilde{\tau} - \eta_\kappa \leqslant \tilde{\tau}(\tilde{\Gamma}_\kappa) \leqslant \tilde{\tau} + \eta_\kappa$.

Note. The optimal shape of the Ising model is obviously square. We expect that all the models of Section 2.1 have an optimal shape, possibly with somehow relaxed condition (ii). [The only property of C in (ii) that will be needed is $\lim_{\kappa \to \infty}(C\kappa^{-1}) = 0$.] We also expect that the optimal shape can be taken such that it respects the symmetries of the Hamiltonian (such as $90°$ rotation in $\mathbb{Z}^\nu$, which is a symmetry present in many concrete Hamiltonians). We do not have proofs of these facts!

Assumption 1. In the following, we study only models having an optimal shape.

As a preliminary step of our constructions, we present here some well-known estimates of the number of lattice points in a given convex set.

Lemma 3. Let $M \subset \mathbb{R}^\nu$ be convex. Denote by $\mathrm{Vol}(M)$ its volume. Denote, for any given $r \in \mathbb{N}$, by $\partial_r^\sim(M)$ the set $\{t \in \mathbb{Z}^\nu\colon \mathrm{dist}(t, M) \leqslant r \ \&\ \mathrm{dist}(t, M^c) \leqslant r\}$. Then:

(i) $(\mathrm{Vol}\,\kappa M)^{-1}\,|M \cap \mathbb{Z}^\nu| \to 1$ as $\kappa \to \infty$.

(ii) $(\mathrm{Vol}\,\kappa M)^{-1}\,|\partial_r^\sim(\kappa M)| \to 0$ as $\kappa \to \infty$.

Next, we will give the definition of contours of the "critical size." Roughly speaking, these will be defined as "optimally shaped" contours Γ^+ satisfying the approximate relation $\tilde{\tau}(\Gamma) \doteq \tilde{\tau}$, such that $\Phi(\Gamma)$ and $aV^*(\Gamma)$ are approximately the same. [It will turn out that $V^*(\Gamma) \doteq (\tilde{\tau}a^{-1})^\nu$ for these contours.]

In the following lemmas, it will be convenient to consider $a > 0$ as a parameter formally not connected with the definition of the quantities $\Phi(\Gamma)$. (See the Note below!)

Lemma 4. There is some $C > 0$ such that for each $\eta > 0$, each $\mu > 0$, and each sufficiently small $a > 0$, there is a contour Γ_μ satisfying the following properties.

(i) $V^*(\Gamma_\mu) \subset \mu\lambda M$, where $\lambda = (\tilde{\tau}a^{-1})(\mathrm{Vol}\,M)^{1/\nu}$.

(ii) $|\tilde{\tau}(\Gamma_\mu) - \tilde{\tau}| \leqslant \eta$.

(iii) $\mathrm{dist}(t, (\mu\lambda M)^c) \leqslant C$ for each $t \in \mu\lambda M \setminus V(\Gamma_\mu)$.

(iv) $|V^*(\Gamma_\mu)| = (\xi\mu)^\nu(\tilde{\tau}a^{-1})^\nu$, with $\xi \to 1$ as $\mu\lambda \to \infty$.

(v) $\Phi(\Gamma_\mu) = \tilde{\xi}\xi^{-1}aV^*(\Gamma_\mu)$ with $|\tilde{\xi} - 1| \leqslant \eta/\tilde{\tau}$.

Proof. Notice first that

$$\mathrm{Vol}(\mu\lambda M) = \mu^\nu(\tau a^{-1})^\nu \tag{42}$$

Taking $\kappa = \mu\lambda$ in the preceding definition and choosing an appropriate $\tilde{\Gamma}_\kappa = \Gamma_\mu$, we have immediately (i)–(iii). To obtain (iv) it suffices to combine Lemma 3 with (42). Finally, (v) follows from (iv) and (ii).

Note. In the formulation of the lemma, $a > 0$ stands like some arbitrary parameter not dependent on the energies $\Phi(\Gamma)$. In fact, the situation in which it will be applied is different, since a depends on $\Phi(\Gamma)$. So, in fact we need a strengthened version of the lemma. So we impose:

Assumption 2. The class of Hamiltonians we study [with e_q and $\Phi(\Gamma)$ depending on some parameters] is such that the estimates of Lemma 4 hold uniformly for all the Hamiltonians, assuming only that the optimal shape M is chosen such that $\mathrm{Vol}(M) = 1$.

Since the optimal shape of the Ising model with an external field is still square, Assumption 2 is satisfied in this particular case. Again, we expect that this assumption is not at all restrictive in usual situations!

Now we can find the "critical set" Λ with the properties 1 and 2 stated in the introduction to this subsection. We will show that it is possible to find a set Λ satisfying 1 somewhere "between" $\frac{3}{4}(1+\varepsilon)\,\lambda M$ and $(1+\varepsilon)\,\lambda M$. Before doing this, we will show in Lemma 5 and Corollary 1 that any Λ "between" $\frac{3}{4}(1+\varepsilon)\,\lambda M$ and $(1+\varepsilon)\,\lambda M$ satisfies the property 2. This will be quite a direct consequence of the very notion of an optimal shape:

Lemma 5. There is some $\xi > 1$ such that $\xi \to 1$ if $\tau \to \infty$, $a \to 0$, and such that for any contour $\Gamma = \Gamma^+$ satisfying the property

$$V^*(\Gamma) \leqslant q(\tilde{\tau} a^{-1})^{\nu}, \qquad q \leqslant 1 \tag{43}$$

the estimate

$$F(\Gamma) \geqslant (1 - \xi q^{1/\nu})\, \Phi(\Gamma) \tag{44}$$

is valid.

Proof. By (3) and by the expression (10) of diluted partition functions we have the estimate

$$\begin{aligned} F(\Gamma) &= \Phi(\Gamma) + \log Z^+(\operatorname{int}_- \Gamma) - \log Z^-(\operatorname{int}_- \Gamma) \\ &\geqslant \Phi(\Gamma) - a\,|V^*(\Gamma)| - C\,|\operatorname{supp} \Gamma| \end{aligned}$$

Choosing some $\tilde{q} > q$, we can write this as

$$F(\Gamma) \geqslant (1 - \tilde{q}^{1/\nu})\, \Phi(\Gamma) - C\,|\operatorname{supp} \Gamma| + \tilde{q}^{1/\nu} \Phi(\Gamma) - a\,|V^*(\Gamma)|$$

Because $C \to 0$ as $\tau \to \infty$, we can write this further as

$$F(\Gamma) \geqslant (1 - \tilde{q}^{1/\nu})\, \Phi(\Gamma) + \tilde{\tilde{q}}^{1/\nu} \Phi(\Gamma) - a\,|V^*(\Gamma)|$$

with a suitable $\tilde{\tilde{q}} < \tilde{q}$ such that $\tilde{\tilde{q}}(\tilde{q})^{-1} \to 1$ as $\tau \to \infty$. Now,

$$\Phi(\Gamma) \geqslant (\tilde{\tau} - \eta)\,|V^*(\Gamma)|^{(\nu-1)/\nu}$$

and therefore

$$\tilde{\tilde{q}}^{1/\nu} \Phi(\Gamma) - a V^*(\Gamma) \geqslant [V^*(\Gamma)]^{(\nu-1)/\nu} [(\tilde{\tau} - \eta)\, \tilde{\tilde{q}}^{1/\nu} - a\,|V^*(\Gamma)|^{1/\nu}]$$

Following the assumption (43), the last bracket is $\geqslant 0$ if $(\tilde{\tau} - \eta)\, \tilde{\tilde{q}}^{1/\nu} > q\tilde{\tau}$. ∎

Corollary 1. Assuming that τ is sufficiently large, ε and a are sufficiently small, any contour Γ satisfying the inequality

$$V^*(\Gamma) \leqslant \tfrac{3}{4}(1+\varepsilon)(\tilde{\tau}a^{-1})^{v}$$

is a small contour.

Proof. This follows from Lemma 5 and the inequality

$$1-\xi[\tfrac{3}{4}(1+\varepsilon)]^{1/v} > 1/4v$$

Recall that $Z_{+}^{\mathrm{small}}(\Lambda)$ denotes the partition function corresponding to all diluted configurations with small external + contours only. Denote by $Z_{+}^{\mathrm{large}}(\Lambda) = Z_{+}(\Lambda) - Z_{+}^{\mathrm{small}}(\Lambda)$.

Note. Corollary 1 says that $Z_{+}^{\mathrm{large}}(\Lambda) = 0$ for the set $\Lambda = V^*(\Gamma_{3(1+\varepsilon)/4})$. Later we will show that $Z_{+}^{\mathrm{large}}(\Lambda)$ "overgrows" the quantity $Z_{+}^{\mathrm{small}}(\Lambda)$ for $\Lambda = V^*(\Gamma_{1+\tilde{\varepsilon}})$. So, as we have noted before, we need to control the growth of Z_{+}^{small} and Z_{+}^{large} with Λ:

Lemma 6. There is some $C = C(\tau)$ such that $\lim_{\tau\to\infty} C = 0$, some $\hat{\tau} > 0$ [depending on the Hamiltonian (1); $\hat{\tau}$ is typically a large quantity] such that:

(i) $|\log Z_{+}^{\mathrm{small}}(\Lambda) - \log Z_{+}^{\mathrm{small}}(\Lambda') - e_{+}\,|\Lambda \triangle \Lambda'|| \leqslant C\,|\Lambda \triangle \Lambda'|$.

(ii) $|\log Z_{+}(\Lambda) - \log Z_{+}(\Lambda')| \leqslant \hat{\tau}\,|\Lambda \triangle \Lambda'|$.

Proof. Condition (i) is a common inequality from the theory of cluster expansions of polymer models, assuming that we write it as

$$|\log Z_{\Lambda} - \log Z_{\Lambda'}| \leqslant C\,|\Lambda \triangle \Lambda'|$$

where Z_{Λ} is the polymer partition function of the contour model with small contours only.

Concerning (ii), it suffices to prove it for $\Lambda' = \Lambda \cup \{t\}$. Take a mapping $\{x_{\Lambda'} \rightsquigarrow x_{\Lambda}\}$ which changes the configuration $x_{\Lambda'}$ "locally" (up to the distance $r+2$ from t) such that dist(supp Γ, $t) \geqslant 2$ for any contour Γ of x_{Λ}. The number of preimages of any x_{Λ} is uniformly bounded by some constant depending on the range of interactions r and also on the number of "spins" $|S|$. Further, for any $x_{\Lambda'}$ we have the obvious inequality [see (1)!]

$$|H(x_{\Lambda'}) - H(x_{\Lambda})| \leqslant \tilde{\tau}$$

where x_{Λ} is the image of $x_{\Lambda'}$ and $\hat{\tau}$ is some constant. So we have (with another $\hat{\tau}$) the inequality (ii).

It implies also the inequality (and this will be used later)

$$|\log Z_{+}^{\mathrm{large}}(\Lambda) - \log Z_{+}(\Lambda')| \leqslant \hat{\tau}\,|\Lambda \triangle \Lambda'| \tag{45}$$

assuming e.g., that $Z_+^{\text{large}}(\Lambda) > Z_+^{\text{small}}(\Lambda)$. This is easily obtained (with a new $\hat{\tau}$) from the relation $Z_+^{\text{large}}(\Lambda) = Z_+(\Lambda) - Z_+^{\text{small}}(\Lambda)$ and (ii).

Note. This is the only point in the paper[2] where the abstract formulation (2) should be supplemented by additional information about the structure of $\Phi(\Gamma)$. Relation (1) gives such information.

Corollary 2. Assume that τ is sufficiently large. Assume that $a > 0$ is sufficiently small. Then there is some Λ satisfying the following inclusions, with suitably small ε, $\tilde{\varepsilon}$:

$$V^*(\Gamma_{3(1+\varepsilon)/4}) \subset \Lambda \subset V^*(\Gamma_{1+\tilde{\varepsilon}}) \tag{46}$$

(see Lemma 4 for the definition of Γ_κ).

Moreover, Λ can be chosen such that

$$0 \geqslant \log Z_+^{\text{small}}(\Lambda) - \log Z_+^{\text{large}}(\Lambda) \geqslant -\hat{\tau} \tag{47}$$

and any large contour Γ in Λ satisfies the relation

$$V^*(\Gamma) \geqslant \tfrac{3}{4}|\Lambda| \tag{48}$$

Proof. We have noted that $Z_+^{\text{large}}(\Lambda') = 0$ for $\Lambda' = V^*(\Gamma_{3(1+\varepsilon)/4})$. In the following we will show that

$$Z_+^{\text{large}}(\Lambda'') > Z_+^{\text{small}}(\Lambda'') \tag{49}$$

for $\Lambda'' = V^*(\Gamma_{1+\tilde{\varepsilon}}) \cup \partial(V^*(\Gamma_{1+\tilde{\varepsilon}}))^c$. Then, by adding successively points from $\Lambda'' \setminus \Lambda'$ to Λ' we obtain, by Lemma 6, (i) and by (45), the existence of a desired set Λ satisfying (46) and (47). Finally, we will deduce from Corollary 1 that any Λ satisfying (46) and any large Γ from Λ satisfy (48).

To prove (49), compare $\log Z_+^{\text{large}}(\Lambda'')$ with $\log Z_+^{\text{small}}$. The former expression is larger than

$$-\Phi_e(\Gamma_{1+\tilde{\varepsilon}}) + \log Z_-(\text{int}_- \Gamma_{1+\tilde{\varepsilon}}) - e_+ \, |\Lambda'' - V^*(\Gamma_{1+\tilde{\varepsilon}})|$$

and so (if we take $\Lambda''' = V^*(\Gamma_{1+\tilde{\varepsilon}})$)

$$\begin{aligned} \log Z_+^{\text{large}}(\Lambda'') &- \log Z_+^{\text{small}}(\Lambda'') \\ &\geqslant -\Phi(\Gamma_{1+\tilde{\varepsilon}}) + a\,|\Lambda'''| - C\,|\text{supp}\,\Gamma_{1+\tilde{\varepsilon}}| \\ &\geqslant -\omega\Phi(\Gamma_{1+\tilde{\varepsilon}}) + a\,|\Lambda'''| \end{aligned}$$

[by Theorem 0, (ii) and (iii); notice that $q = -$ is stable!], where $\omega > 1$ can be chosen such that $\lim_{\tau\to\infty} \omega = 1$. By Lemma 4, (v) this is $\geqslant 0$ for large τ.

[2] Another point is the proof of completeness.[(2)]

This proves (47). By Corollary 1, for any large Γ we have $V^*(\Gamma) > \frac{3}{4}(1+\varepsilon)(\tilde{\tau}a^{-1})^{\nu}$, but $(1+\varepsilon)(\tilde{\tau}a^{-1})^{\nu} \geqslant |\Lambda''| \geqslant |V^*(\Gamma_{1+\tilde{\varepsilon}})|$ (for a suitable $\tilde{\varepsilon} > 0$) because of Lemma 4, (iv). So $(1+\varepsilon)(\tilde{\tau}a^{-1})^{\nu} \geqslant |\Lambda|$ and $V^*(\Gamma) \geqslant \frac{3}{4}|\Lambda|$. This completes the proof of Corollary 2.

3.2. The Case of Complex Hamiltonians

Assume now that the real Hamiltonian H, for which all the preparatory constructions of Section 3.1 (notably, Corollary 2) have been done, is a particular value $H(0) = H$ of a family of complex Hamiltonians $\{H(\lambda), \lambda \in \vartheta\}$, where ϑ is some complex neighborhood of zero. More specifically, we will assume that the quantities $e_q(\lambda)$ and $\Phi(\Gamma, \lambda)$ change linearly in λ as follows, with real $\tilde{e}_q$ and $\tilde{\Phi}(\Gamma)$:

$$e_q(\lambda) = e_q + \lambda \tilde{e}_q, \qquad \Phi(\Gamma, \lambda) = \Phi(\Gamma) + \lambda \tilde{\Phi}(\Gamma) \tag{50}$$

where the quantities $\tilde{e}_q$ (recall that $Q = \{+, -\}$) and $\tilde{\Phi}(\Gamma)$ satisfy

$$\tilde{e}_+ - \tilde{e}_- = 1 \tag{51}$$

and some bound of the type

$$|\tilde{\Phi}(\Gamma)| \leqslant K\,|\text{supp}\,\Gamma| \tag{52}$$

with K to be specified later.

Note. Condition (52) implies another bound,

$$\left|\frac{d}{d\lambda}\exp[-\Phi(\Gamma, \lambda)]\right| \leqslant \exp(-\tau'\,|\text{supp}\,\Gamma|) \tag{52'}$$

with a large τ' (assuming that K is not too large). This more general bound is used in Ref. 7, but here it would complicate our estimates, so we do not use it.

From now on, take the set Λ as specified in Corollary 2. As in Section 2, our strategy is to *fix* the notion of a small contour when defining the partition functions $Z_+^{\text{small}}(\Lambda, \lambda)$, etc., for the complex Hamiltonian $H(\lambda)$. (This will require a guarantee of the applicability of cluster expansions to $Z_+^{\text{small}}(\Lambda, \lambda)$—see Lemma 7).

Consider the analytic functions

$$f(\lambda) = Z_+^{\text{small}}(\Lambda, \lambda), \qquad g(\lambda) = Z_+^{\text{large}}(\Lambda, \lambda)$$

Put

$$\varphi(\lambda) = \log f(\lambda) - \log g(\lambda)$$

Statement (47) also can be formulated as follows:

$$0 \geq \varphi(0) \geq -\tilde{\tau} \quad \text{(with a new } \hat{\tau}\text{)} \tag{53}$$

We will show that $\varphi(\lambda) = -i\pi$, i.e., $f(\lambda) = -g(\lambda)$ for a suitable λ:

Theorem 2. Assume that $a > 0$ is sufficiently small, τ sufficiently large, and K not too large (to be specified later). Then there is some $\lambda \in \mathbb{C}$ such that

$$|\mathrm{Re}\,\lambda| \leq 2\hat{\tau}\,|\Lambda|^{-1} \tag{54}$$

$$|\mathrm{Im}\,\lambda - \pi\,|\Lambda|^{-1}| \leq 2\,|\Lambda|^{-1} \tag{55}$$

such that $\varphi(\lambda) = i\pi$ [i.e., such that $Z_+(\Lambda, \lambda) = 0$]. ($\hat{\tau}$ is from (53)).

Note. Consider the case of the Ising model with a positive external field $\frac{1}{2}\lambda$. Then a behaves like $\beta\lambda$ where β is the inverse temperature. The bounds (55) resp. (54) are of the order λ^ν resp. $\beta\lambda^\nu$ because both $\hat{\tau}$ and $\tilde{\tau}$ are proportional to the inverse temperature.

The *proof* of Theorem 2 will be divided into two steps. In the second step we will find a real Hamiltonian $H(\lambda_0)$ satisfying (54) and the relation

$$\varphi(\lambda_0) = 0 \tag{56}$$

[This is a suitable sharpening of (47) only].

In the first step of the proof we will show that assuming (56) for $\lambda_0 = 0$, we can find some $H(\lambda)$ satisfying $\varphi(\lambda) = -i\pi$ such that $|\lambda + \pi i\,|\Lambda|^{-1}| < 2\,|\Lambda|^{-1}$. This is the more characteristic part of the proof and some of the techniques developed here will be used also in the second step.

As a preparation, we need a lemma, which gives a suitable sharpening of (27).

Lemma 7. Let λ, λ_0 be complex parameters such that the inequality (e.g.)

$$\mathrm{Re}\,F(\Gamma, \tilde{\lambda}) \geq (\tau/5\nu)\,|\mathrm{supp}\,\Gamma| \tag{57}$$

holds for each small $\Gamma = \Gamma^+$ and each $\tilde{\lambda}$ from the interval $[\lambda_0, \lambda]$. Then

$$\mathrm{Re}\,F(\Gamma, \lambda) \geq \mathrm{Re}\,F(\Gamma, \lambda_0) - 2\,|\lambda - \lambda_0|\,|V^*(\Gamma)| - K\,|\lambda - \lambda_0|\,|\mathrm{supp}\,\Gamma| \tag{58}$$

where K is taken from (52).

Proof. From the definition (3) we have

$$F(\Gamma, \lambda) = \Phi(\Gamma, \lambda) + \log Z_+(\text{int}_- \Gamma, \lambda) - \log Z_-(\text{int}_- \Gamma, \lambda) \tag{59}$$

and so we have to estimate how the terms on the right-hand side of this relation change with λ.

The estimate of $\Phi(\Gamma, \lambda)$ follows immediately from (52):

$$\frac{d}{d\lambda} \Phi(\Gamma, \lambda) = \tilde{\Phi}(\Gamma) \geqslant -K\,|\text{supp}\,\Gamma| \tag{60}$$

Concerning $Z_+(\text{int}_- \Gamma, \lambda)$ and $Z_-(\text{int}_- \Gamma, \lambda)$, we will estimate them at the same time as $F(\Gamma, \lambda)$. Namely, we will prove the following estimates ($q = +$ or $-$):

$$\left| \frac{d}{d\lambda} [\log Z_q(M, \lambda) - e_q(\lambda)\,|M|] \right| \leqslant \eta\,|M| \tag{61}$$

together with the estimate [which is a stronger version of (58)]

$$\left| \frac{d}{d\lambda} F(\Gamma, \lambda) \right| \leqslant (1 + 2\eta)\,|V^*(\Gamma)| + K\,|\text{supp}\,\Gamma| \tag{62}$$

where $\eta = \eta(K, \tau)$ is such that $\lim_{\tau \to \infty} \eta = 0$. It is obvious that (61) and (60) substituted into (59) yield (62). To show that also (62) implies (61), consider the expression

$$Z_q(M, \lambda) = \exp[e_q(\lambda)\,|M|]\, Z^q_M(\lambda) \tag{63}$$

where $Z^q_M(\lambda)$ is the polymer partition function corresponding to all (small) Γ^q and the activity $k_{\Gamma^q} = \exp[-F(\Gamma^q, \lambda)]$. Expressing further $(d/d\lambda) \log Z^q_M(\lambda)$ as

$$\frac{d}{d\lambda} \log Z^q_M(\lambda) = \sum p(\Gamma^q) \frac{d}{d\lambda} \exp[-F(\Gamma^q, \lambda)]$$

$$p(\Gamma^q) = Z^q_{M \setminus \text{supp}\,\Gamma}(\lambda) [Z^q_M(\lambda)]^{-1}$$

and noticing that

$$\exp[-F(\Gamma^q, \lambda)] \leqslant \exp[(-\tau/5\nu + \varepsilon)\,|\text{supp}\,\Gamma|], \qquad \varepsilon \text{ small}$$

we obtain (61) from (62), which concludes the inductive proof of these relations, and also the proof of (58).

Now we can go to the proof of Theorem 2. We will study the quantities $f(\lambda)$ and $g(\lambda)$ and the relation between them. Investigate first $f(\lambda)$ by expressing it as follows:

$$f(\lambda) = Z_{+}^{\text{small}}(\Lambda, \lambda) = \exp[-h_{+}(\lambda)\,|\Lambda| + \Delta(\lambda)]$$
$$= \exp[-h(\lambda)\,|\Lambda|]\,\exp[-a(\lambda)\,|\Lambda|]\,\exp[\Delta(\lambda)] \tag{64}$$

where $|\Delta(\lambda)| \leqslant \bar{\varepsilon}\,|\partial\Lambda|$, $\bar{\varepsilon} \to 0$, for $\tau \to \infty$. Moreover, by (38), $|\Delta(\lambda) - \Delta| \leqslant \tilde{\varepsilon}\,|\lambda|\,|\partial\Lambda|$ where $\tilde{\varepsilon} \to 0$ for $\tau \to \infty$. Therefore, the leading term in

$$\tilde{f}(\lambda) = f(\lambda)\exp[h(\lambda)\,|\Lambda|]$$

is $\exp[-a(\lambda)\,|\Lambda|]$ because $\Delta(\lambda)$ can be "neglected" for large $|\Lambda|$ (i.e., for small $a > 0$).

On the other hand, the term

$$\tilde{g}(\lambda) = g(\lambda)\exp[h(\lambda)\,|\Lambda|]$$

has a different behavior. It changes itself much more slowly: Investigate [the summation is over all systems $\{\Gamma_i\}$ of large + contours in Λ with disjoint $V^*(\Gamma_i)$, $\text{ext} = \Lambda \backslash \bigcup_i V^*(\Gamma_i)$]

$$g(\lambda) = \sum_{\{\Gamma_i\}} \exp\left[-\sum_i \Phi_e(\Gamma_i, \lambda)\right] Z_{+}^{\text{small}}(\text{ext}) \prod_i Z^{-}(\text{int}_{-}\,\Gamma_i)$$
$$= \sum_{\{\Gamma_i\}} \exp\left[-\sum_i \Phi(\Gamma_i, \lambda) - h_{+}(\lambda)\,|\text{ext}| + h(\lambda)\,|\Lambda\backslash\text{ext}| + \tilde{\Delta}(\lambda)\right] \tag{65}$$

Write it as

$$\tilde{g}(\lambda) = \tilde{g}(0)\,E(\xi_{\mathscr{D}}\theta_{\mathscr{D}}\eta_{\mathscr{D}}) \tag{66}$$

where

$$\xi_{\mathscr{D}} = \exp\left[-\sum_i \Phi(\Gamma_i, \lambda) + \Phi(\Gamma_i)\right]$$
$$\theta_{\mathscr{D}} = \exp\{-[a(\lambda) - a]\,|\text{ext}|\}$$
$$\eta_{\mathscr{D}} = \exp[\tilde{\Delta}(\lambda) - \tilde{\Delta}]$$

and the expectation is taken with respect to the probability

$$P(\mathscr{D}) = \exp\left[-\sum \Phi(\Gamma_i) - a\,|\text{ext}| + \tilde{\Delta}\right]\tilde{g}(0)^{-1}$$

assigned to any system of large contours $\mathscr{D} = \{\Gamma_i\}$.

We will show that for any λ from the disk $|\lambda\,|\Lambda|-\pi i|\leqslant 2$,

$$|\log \tilde{g}(\lambda)\, \tilde{g}(0)^{-1}| = |\log E(\xi_{\mathscr{D}}\theta_{\mathscr{D}}\eta_{\mathscr{D}})| \leqslant 3/2 \tag{67}$$

(for the special value of $\lambda = \pi i\,|\Lambda|^{-1}$ we will need a more precise inequality).

Consider first the "most important" quantity $\theta_{\mathscr{D}}$. Since $|\text{ext}| \leqslant \frac{1}{4}|\Lambda|$ and $a(\lambda)[e_+(\lambda)-e_-(\lambda)]^{-1} \to 1$ for $\tau \to \infty$, we may assume that

$$|a(\lambda)-a|\;|\text{ext}| \leqslant \tfrac{1}{4}|\Lambda|\;|a(\lambda)-a| \leqslant \tfrac{1}{4}C(\tau)\,|\Lambda|$$

where $C(\tau) \to 1$ as $\tau \to \infty$. Notice that

$$e_+(\lambda)-e_-(\lambda)-e_+ + e_- = \lambda(\tilde{e}_+ - \tilde{e}_-) = \lambda$$

So

$$\theta_{\mathscr{D}} = \exp(\tfrac{1}{4}\tilde{\theta}\lambda\,|\Lambda|) \tag{68}$$

where $|\text{Re}\,\tilde{\theta}| \leqslant C(\tau)$, $|\text{Im}\,\tilde{\theta}| \leqslant \tilde{C}(\tau)$ and where $C(\tau)$ and $\tilde{C}(\tau)$ can be chosen such that $\lim_{\tau\to\infty} C(\tau)=1$ and $\lim_{\tau\to\infty} \tilde{C}(\tau)=0$.

Concerning the quantity $\xi_{\mathscr{D}}$, we use the following consideration. For large Λ, $|\partial\Lambda|$ is very small compared to $|\Lambda|$ (because Λ has an "optimal shape") and so a "typical value" [counted in $P(\mathscr{D})$] of $\sum |\text{supp}\,\Gamma_i|$ is also very small compared to $|\Lambda|$. More precisely, we have

$$E(\xi_{\mathscr{D}}-1)^2 \to 0 \tag{69}$$

if $a \to 0$ (uniformly in $|2\,|\Lambda| - \pi_i| \leqslant 2$).

A similar consideration can be applied to $\eta_{\mathscr{D}}$. The novel point here is that we have no *a priori* estimate (52). An analogical inequality

$$|\tilde{\Delta}(\lambda)-\tilde{\Delta}(\lambda_0)| \leqslant C\,|\lambda-\lambda_0| \sum_i |\text{supp}\,|\Gamma_i|| \tag{70}$$

can be proved, however, as we explained in (38).

Thus, we have also

$$E(\eta_{\mathscr{D}}-1)^2 \to 0 \qquad \text{for} \quad a \to 0 \tag{71}$$

Finally, taking into account (71), (69), and (68), we obtain, for sufficiently large τ and sufficiently small a, the inequalities

$$\text{Re}\,E(\xi_{\mathscr{D}}\theta_{\mathscr{D}}\eta_{\mathscr{D}}) \geqslant \exp\left(-\tilde{\theta}\,|\Lambda|\,\frac{|\text{Re}\,\lambda|}{4}\right)\cos\left(\tilde{\tilde{\theta}}\,|\Lambda|\,\frac{\text{Im}\,\lambda}{4}\right) \tag{72}$$

and

$$|E(\xi_{\mathscr{D}}\theta_{\mathscr{D}}\eta_{\mathscr{D}})| \leqslant \exp\left(\tilde{\tilde{\tilde{\theta}}}\,|\Lambda|\frac{|\lambda|}{4}\right) \tag{73}$$

with some real $\tilde{\theta}$, $\tilde{\tilde{\theta}}$, and $\tilde{\tilde{\tilde{\theta}}}$ converging to 1 for $\tau \to \infty$ and $a \to 0$. Consider the quantity

$$m = \min\{f(x, y); x^2 + y^2 \leqslant 4\}$$

where $f(x, y) = \exp(-x/4)\cos[(\pi + y)/4]$. We have the simple estimate $m > m'$, where

$$\begin{aligned} m' &= \min\{f(x, y); x \geqslant 0, 2 \geqslant y \geqslant 0, x + y \leqslant 2\sqrt{2}\} \\ &= \exp\left(\frac{\sqrt{2}-1}{2}\right)\cos\left(\frac{\pi+2}{4}\right) > \exp\left(-\frac{3}{2}\right) \end{aligned}$$

So we obtain from (72) the inequality

$$\operatorname{Re} E(\xi_{\mathscr{D}}\theta_{\mathscr{D}}\eta_{\mathscr{D}}) > \exp(-1.5) \tag{74}$$

for any sufficiently large τ and any sufficiently small a. This can be complemented by another set of numerical bounds

$$|\operatorname{Im} E(\xi_{\mathscr{D}}\theta_{\mathscr{D}}\eta_{\mathscr{D}})| < 1.5$$

and

$$|E(\xi_{\mathscr{D}}\theta_{\mathscr{D}}\eta_{\mathscr{D}}) - 1| < 2.2 \tag{75}$$

which are obtained in an analogous way, estimating the functions $\exp(-x/4)\sin((\pi+y)/4)$ resp. $\exp(-x/4 + i(\pi+y)/4) - 1$ on the disk $x^2 + y^2 \leqslant 4$. All these bounds hold for $|\lambda\,|\Lambda| - \pi i| \leqslant 2$ and they finally give (67) after some computations (namely, $\operatorname{Re} z \geqslant \exp(-1.5)$ together with $|\operatorname{Im} z| \leqslant 1.5$ and $|z - 1| < 2.2$ implies that $|\log z| \leqslant 1.5$).

For λ almost equal to $\pi i\,|\Lambda|^{-1}$ a more precise inequality can be obtained: Let $\lambda_\delta = i\,|\Lambda|^{-1}(\pi - \delta)$ with $-1 \leqslant \delta \leqslant 1$. Then (for a suitable $\varepsilon > 0$ such that $\lim_{\tau \to \infty, a \to 0} \varepsilon = 0$)

$$\operatorname{Re} \tilde{g}(\lambda_\delta)\,\tilde{g}(0)^{-1} > 2^{(1/2)} - \varepsilon$$

and

$$|\tilde{g}(\lambda_\delta)\,\tilde{g}(0)^{-1}| < 1 + \varepsilon$$

This implies (with another ε) that

$$|\operatorname{Re}\log \tilde{g}(\lambda_\delta)\,\tilde{g}(0)^{-1}| \leqslant 0.35 + \varepsilon \tag{76}$$

and

$$|\operatorname{Im}\log \tilde{g}(\lambda_\delta)\,\tilde{g}(0)^{-1}| < 1$$

Compare now these inequalities with (64). Because $a = a(\lambda) = e_+(\lambda) - e_-(\lambda) + \tilde{\varepsilon} = \lambda + \tilde{\varepsilon}$ with some $\tilde{\varepsilon} \ll \lambda$ (see (37)) we have

$$\tilde{f}(\lambda) = \tilde{f}(0) \exp((-\lambda - \tilde{\varepsilon})\,|\Lambda|) \tag{77}$$

(with another $\tilde{\varepsilon}$).

Consider the function (rescaled φ)

$$\begin{aligned}\hat{\varphi}(\lambda) &= \log f(\lambda\,|\Lambda|^{-1}) - \log g(\lambda\,|\Lambda|^{-1}) + \pi i \\ &= \log \tilde{f}(\lambda\,|\Lambda|^{-1}) - \log \tilde{g}(\lambda\,|\Lambda|^{-1}) + \pi i\end{aligned}$$

Recall that we want to solve the equation $\hat{\varphi}(\lambda) = 0$. An immediate consequence of (67) and (77) is the inequality

$$|\hat{\varphi}(\lambda)| \geqslant 2 - 1.5 - |\tilde{\varepsilon}|$$

which holds on the circle $|\lambda - \pi i| = 2$. On the other hand, by (76) and (77) we have for some $-1 \leqslant \delta \leqslant 1$ the estimate

$$|\hat{\varphi}[(\pi - \delta)i]| \leqslant 0.35 + \varepsilon + |\tilde{\varepsilon}|.$$

This means, however, that $\hat{\varphi}[(\pi - \delta)i]^{-1} > \max\{|\hat{\varphi}(\lambda)|^{-1}, |\lambda - \pi_i| \leqslant 2\}$ and so the function $\hat{\varphi}(\lambda)^{-1}$ cannot be holomorphic. In other words, the equation $\hat{\varphi}(\lambda) = 0$ must have solution in the disk $|\lambda - \pi_i| \leqslant 2$. This concludes the first step of the proof of Theorem 2.

It remains to be shown that there is some real λ_0 such that (56) holds. This can be easily shown by modifying the estimates of $\tilde{f}$ and $\tilde{g}$ given above. Instead of (68) we have for any real $\lambda > 0$ the estimate

$$|\theta_{\mathscr{D}}| \leqslant \exp\left[\left(\frac{\lambda}{4} + \varepsilon\right)|\Lambda|\right], \qquad \varepsilon \ll \lambda$$

Substituting this into (66) (together with (69) and (71)) we have, with another $\varepsilon \ll \lambda$, the estimate

$$\tilde{g}(\lambda) \leqslant \tilde{g}(0) \exp\left[\left(\frac{\lambda}{4} + \varepsilon\right)|\Lambda|\right] \tag{78}$$

So, if we combine it with (77),

$$\tilde{f}(\lambda)\,\tilde{g}(\lambda)^{-1} \geqslant \exp[(-\tfrac{3}{4}\lambda - \varepsilon - |\tilde{\varepsilon}|)\,|\Lambda|]\,\tilde{f}(0)\,\tilde{g}(0)^{-1}$$

Recalling that $\tilde{f}(0)^{-1}\tilde{g}(0) < \exp(\hat{\tau})$ we surely get, for some $0 > \lambda_0 > -2\,|\Lambda|^{-1} \log \hat{\tau}$, the desired solution of the equation

$$\tilde{f}(\lambda_0) = \tilde{g}(\lambda_0)$$

which completes the proof of Theorem 2.

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